# Lecture 12.5: Review for Midterm

- Topics:

PDEs

- Classical Solution
- Equation types:

  linear

  - Elliptic — Laplace
  - Parabolic — Heat
  - Hyperbolic — Wave
- Well-posedness

  existence, uniqueness, continuous dependence on data

Preliminaries

- $\mathbb{R}$ & $\mathbb{R}^n$, supremum/limits
- $\mathbb{C}$, complex conjugation, $e^{i\theta}$
- Neighborhoods, open/closed, boundary, connectedness, boundedness, compactness.
- Differentiability – $C^m$, $C_c^m$, support, smooth bumps
- ODEs – Separation of variables, Auxilliary equations, Picard-Lindelöf.
- Vector calculus – gradient, $\Delta = \nabla \cdot \nabla$, surface integration, radial Laplacian, Divergence thm. $\int_\Omega \nabla \cdot F = \int_{\partial\Omega} F \cdot n \, dS$

  If $F = \nabla u$, $\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \partial u/\partial n \, dS$

  $\rightsquigarrow$ Green's Identity $\int_\Omega \nabla v \cdot \nabla u + v \Delta u \, dx = \int_{\partial\Omega} v \, \partial u/\partial n \, dS$

# Conservation Equations & Characteristics

→ $\partial u/\partial t + v\cdot\nabla u + (\nabla\cdot v)u = 0$

or $\quad \partial u/\partial t + v\cdot\nabla u + w = 0$

Set $\frac{dx}{dt} = v \qquad \& \qquad \frac{Du}{Dt} = \frac{d}{dt}u(t, x(t))$.

$\Rightarrow \quad \frac{Du}{Dt} + w(t, x(t), u(x(t))) = 0$

~quasilinear case: $\partial u/\partial t + a(u)\cdot\nabla u = 0$

$\rightsquigarrow \quad x(t) = x_0 + a(u(0,x_0))t$ characteristics.

→ Shock Formation, Burgers' Equation.

# Wave Equation

$$\begin{cases} u(0,x) = g(x) \\ \partial_t u(0,x) = h(x) \end{cases}$$

$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ or $\frac{\partial^2 u}{\partial t^2} - c^2\Delta u = 0$

1D: Solution by characteristics

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)d\tau$$

- left & right wave parts
- Huygens' Principle: $\mathrm{Supp}(g), \mathrm{supp}(h) \subseteq [a,b]$

$\Rightarrow \mathrm{Supp}(u) \subset \{(t,x) : x \in [a-ct, b+ct]\}$

Boundary Problems - on $[0,\ell]$

$h \& g$ should extend to odd, $C^2$, $2\ell$-periodic functions

Forcing term $\quad \partial^2 u/\partial t^2 - c^2 \partial^2 u/\partial x^2 = f, \quad u(0,x) = 0 = \partial_t u(0,x)$

$D_{t,x} = \{(s,y) : x - c(t-s) \le y \le x + c(t-s)\}$

$u(t,x) = \frac{1}{2c}\int_{D_{t,x}} f(s,y)\,ds\,dy$

- Superposition for mixed problem

Higher Dimensions - Odd dimensions by spherical symmetry,

Huygens' gives a "sharp wave cone"

$\Gamma_+(t_0,x_0) = \{(t,x) : t > t_0, \ |x - x_0| = t - t_0\}$

(Kirchhoff for 3D)

- Even dimensions by the method of descent (Poisson for 2D)
  ~ Wave cone isn't sharp: a sudden disturbance has a lingering tail

Energy methods
$\int |\nabla_{t,x} u|^2 dx$ constant for wave solutions $\Rightarrow$ Unique Solutions

# Separation of Variables

Product Solutions $u(t,x) = v(t)\phi(x)$

$(\partial_t - \Delta)u = 0$ splits to $\partial_t v = \kappa v$, $\Delta\phi = \kappa\phi$ for $\kappa \in \mathbb{R}$

Helmholtz Problem

$-\Delta\phi = \lambda\phi$ ⇝ 1D is easy, overtones

Higher dimensions are complicated, rely on symmetries